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Hilbert flag varieties and their Kähler structure

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Abstract
In this paper, we introduce the infinite-dimensional flag varieties associated with integrable systems of the KdV- and Toda-type and discuss the structure of these manifolds. As an example we treat the Fubini–Study metric on the projective space associated with a separable complex Hilbert space and conclude by showing that all flag varieties introduced before possess a Kähler structure.

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1. Introduction

Apart from the fact that flag varieties play a central role in the representation theory of Lie groups, they are also prominent geometric objects in various parts of mathematical physics. In the finite dimensional case, e.g., a double fibration of flag varieties is the central object in the Penrose transform and its generalizations (see [BE89]).

For soliton equations of KdV-type, infinite dimensional Grassmannians form the geometric structure, see e.g. [SS83, and [SW85], that yields solutions of these equations and the towers of equations, so-called hierarchies, linked to them. Associated with these planes are the so-called τ-functions in which the solutions of the soliton hierarchies can be expressed. These functions can be given a geometric description in terms of the determinant bundle over this variety. In the same context finite chains of subspaces in the Grassmann manifold correspond to so-called modified equations of these hierarchies and to Darboux transformations (see resp. [HH94] and [HvdL01a]). Infinite chains of subspaces occur naturally at Toda-type hierarchies, as one can read off from, e.g., [AHvM93] and [FH91].

These varieties also turn up in quantum field theory. We mention a few instances. For example, in quantum field theory in two space–time dimensions, the possible boundary conditions for the Dirac operator give you a point in a Grassmannian (see [Wit88]). By using the structure of the determinant bundle over this variety, Witten arrived at so-called multiplicative Ward identities. In [KNTY88], the relevant moduli spaces for conformal field theories over Riemann surfaces were embedded in a Grassmannian, which enabled an explicit calculation of *a priori* provided quantities, such as the current and energy–momentum tensor, in terms of the moduli. Moreover, it led to a characterization of the image by a system of differential equations for the corresponding τ-functions. From a Grassmannian description of the theory Fukuma, Kawai and Nakayama deduced in [FKN92] the $W_{1+\infty}$-symmetry of two-dimensional quantum gravity and obtained the reduction of the tower of constraints to the lowest level. Also the conjectures of Witten that link the partition function of two-dimensional quantum gravity to τ-functions of integrable hierarchies support the central character of this geometric structure.

The foregoing examples underline the importance of a good knowledge of the structure of infinite flag varieties. Here we present an analytic category of flags that contains the Grassmann manifolds of all subspaces of a fixed finite dimension and that is at the same time a natural extension of the Grassmannians from [SW85]. We describe their manifold structure and show that they all possess a Kähler structure. This last fact can be used at the Hamiltonian description of the integrable systems alluded to above.

The precise content of the various sections is as follows: the second section describes the type of flags that we will consider and gives two infinite dimensional Lie groups that act transitively on the space $\mathfrak{F}$ of these flags. In the third section we present useful decompositions of a number of open subsets in these Lie groups and the manifold structure of $\mathfrak{F}$. The connected components of the variety $\mathfrak{F}$ are described in the fourth section. Next we treat as an example the Fubini–Study metric on projective Hilbert space. The final section is devoted to the description of the Kähler structure on $\mathfrak{F}$.

2. The flag variety

First we will discuss the form of the flag varieties that will be considered. We start out with a separable complex Hilbert space H equipped with an inner product $\langle\cdot,\cdot\rangle$ and a decomposition

$$H = \bigoplus_{-N-1<i<M+1} H_i \qquad \text{where} \quad H_i \perp H_j \text{ for } i \neq j \text{ and } H_i \text{ is closed.} \tag{1}$$

Here, as in the rest of this paper, the direct sum of a number of Hilbert spaces is meant to be the Hilbert direct sum, as soon as it is infinite. This will be assumed from now on without further mention. We have no restriction on the dimension of the spaces H_i, in other words $m_i = \dim(H_i)$ satisfies $1 \leqslant m_i \leqslant \infty$. For convenience, we write $I = \{j \mid -N-1 < j < M+1\}$ for the collection of indices of the decomposition (1). With respect to the decomposition (1) we consider for each i, $i \in I$, the subspace

$$H(i) = \bigoplus_{-N-1<j\leqslant i} H_j \tag{2}$$

and this gives rise to the so-called *basic flag* $F^{(0)}$, which will be the leading example of a flag and which consists of the chain of subspaces

$$H(-N-1) = 0 \ldots \subset H(i-1) \subset H(i) \subset \ldots \subset H = H(M).$$

More generally, we consider inside H flags $F = \{F(i) \mid i \in I\}$, which is to say chains of closed subspaces of H,

$$\{0\} = F(-N-1) \ldots \subset F(i) \subset F(i+1) \ldots \subset F(M) = H.$$

With any such flag F is associated an orthogonal decomposition of H,

$$H = \bigoplus_{i\in I} F_i \qquad \text{where} \qquad F_i = F(i) \cap F(i-1)^{\perp}.$$

We will denote the flag F both by $F = \{F(i) \mid i \in I\}$ as well as by $F = \{F_i \mid i \in I\}$. The choice of the Hilbert space and the relevant decomposition depends on the process one wants to describe as the following examples show.

Example 2.1. In quantum field theory the states of the system are vectors in a Fock space. In the fermionic case, this space is built up from the splitting of a basic Hilbert space into positive and negative energy states. For example, at the Dirac theory for a one-dimensional particle of mass $m \geqslant 0$, one considers, see [CR87], the Dirac Hamiltonian

$$D = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}.$$

It acts on the Hilbert space $H = L^2(\mathbb{R})^2$ and the relevant decomposition of H for the Fock space representation is $H = H_+ \oplus H_-$, where H_+ is the subspace of H corresponding to the positive spectrum of D and H_- the one corresponding to the negative spectrum of D. In general, see [Mick], Dirac operators are associated with a number of geometric data, such as a spin bundle over an oriented Riemannian manifold, another vector bundle over this manifold and a connection. A similar splitting of the square-integrable extended Dirac spinors is the starting point for the construction of a representation of an extension of the group of gauge transformations (see [Mick]).

Example 2.2. In the case of the KP-theory and its many subsystems, the relevant space H is the Hilbert space

$$L^2(S^1, \mathbb{C}) = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n, a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty \right\}$$

with the inner product

$$\left\langle \sum_{n\in\mathbb{Z}} a_n z^n \,\middle|\, \sum_{n\in\mathbb{Z}} b_n z^n \right\rangle = \sum_{n\in\mathbb{Z}} a_n \bar{b}_n.$$

Decompositions that play a role in this context can be described as follows: consider a finite M and N and let $\underline{s} = (s_{-N}, \ldots, s_{M-1})$, where $s_i \in \mathbb{Z}$ and $s_{i+1} < s_i$. Then we take the basic flag defined by

$$H(i) = \left\{ \sum_{n\geqslant s_i} a_n z^n \in H \right\} \qquad -N \leqslant i \leqslant M-1$$

which is complemented with $H(-N-1) = \{0\}$ and $H(M) = H$. For matrix versions of the KP-hierarchy, one uses the Hilbert space $L^2(S^1, \mathbb{C}^n)$ with similar decompositions.

The class of flags under consideration here, have, first of all, the same length as the basic flag $F^{(0)}$. Secondly, the components of the flag have the same 'size' as the corresponding components of $F^{(0)}$, i.e. for all $i, i \in I$,

$$\dim(F_i) = \dim(H_i)$$

and finally we require our flags not to differ too much from the basic flag. This last property is specified in the definition below. Before that we introduce some notation. Let $p_i, i \in I$, be the orthogonal projection of H onto H_i. Decomposition (1) induces also one for linear operators on H.

Notation 2.3. If g belongs to $\mathcal{B}(H)$, the space of bounded linear operators from H to H, then $g = (g_{ij})$, $i \in I$ and $j \in I$, denotes the decomposition of g w.r.t. the $\{H_i \mid i \in I\}$. That is to say $g_{ij} = p_i \circ g \mid H_j$. If $g \in \mathcal{B}(H)$ and $i_0, i_0 \in I$, then we write suggestively $g_{\geqslant i_0}$ for $\sum_{i \geqslant i_0} p_i g \mid \oplus_{i \geqslant i_0} H_i$ in $\mathcal{B}(\oplus_{i \geqslant i_0} H_i)$ and $g_{\leqslant i_0}$ for $\sum_{i \leqslant i_0} p_i g \mid \oplus_{i \leqslant i_0} H_i$ in $\mathcal{B}(\oplus_{i \leqslant i_0} H_i)$. Finally, if we have operators $u_i \in \mathcal{B}(H_i)$, $i \in I$, then we write

$$\mathrm{diag}(u_i) := \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & u_{i+2} & 0 & 0 & \ddots \\ \ddots & 0 & u_{i+1} & 0 & \ddots \\ \ddots & 0 & 0 & u_i & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Definition 2.4. *Let $\mathfrak{F}$ be the collection of flags $F = \{F_i \mid i \in I\}$, satisfying three properties. First of all, the equality* $\dim(F_i) = \dim(H_i)$ *should hold for all $i \in I$. Next the operator $\sum_i \sum_{j \neq i} p_j \mid F_i : H \to H$ is required to be a Hilbert–Schmidt operator. Finally, the operator $\sum_{i \in I} p_i \mid F_i$ in $\mathcal{B}(H)$ that maps each subspace F_i to H_i should be a Fredholm operator of order zero. We call $\mathfrak{F}$ the flag variety corresponding to the decomposition* (1).

Remark 2.5. If there is only a finite number of components in the decomposition (1), then these flag varieties are exactly the ones introduced in [HH94]. The need for infinite chains of subspaces comes up naturally in the context of Toda-type hierarchies (see, e.g. [HvdL01b]). If there is only a finite number of components and only one m_i is infinite, then the Hilbert–Schmidt condition is superfluous. Hence the present class of flag varieties includes both the Grassmann variety $Gr(k, H)$ of all k-dimensional subspaces of H as well as the Grassmann variety $Gr(H)$ introduced by Pressley and Segal in [PS86]. As is well known by now, this last manifold is the Hilbert variety from which a rich collection of solutions for the KP-hierarchy could be constructed. The flag varieties of finite length in this last category occur naturally in the context of the KP-hierarchy when considering modified equations and Darboux transformations (see, e.g., [HH94] and [HvdL01a]).

Remark 2.6. Since $H = \oplus_i H_i = \oplus_i F_i$, the condition that the operator $\sum_i p_i \mid F_i$ is Fredholm of index zero implies that each $p_i \mid F_i$ is Fredholm and that the index of all but a finite number of these Fredholm operators is zero.

Like the finite dimensional flag varieties the space $\mathfrak{F}$ is also a homogeneous space for several groups. First of all there is the analogue for $\mathfrak{F}$ of the general linear group. Its Lie algebra is formed by

Definition 2.7. *A restricted endomorphism of H is a $u = (u_{ij})$ in $\mathcal{B}(H)$ such that $u - \mathrm{diag}(u_{ii})$ is a Hilbert–Schmidt operator. We denote the space of all restricted endomorphisms of H by $\mathcal{B}_{\mathrm{res}}(H)$.*

The algebra $\mathcal{B}_{\mathrm{res}}(H)$ becomes a Banach algebra if we equip it with the norm $\|\cdot\|_2$ defined by

$$\|u\|_2 = \|u\| + \sum_{i \neq j} \|u_{ij}\|_{\mathcal{HS}}$$

where $\|.\|$ denotes the operator norm and $\|.\|_{\mathcal{HS}}$ the Hilbert–Schmidt norm. If one chooses an orthogonal basis $\{e_k(i) \mid 0 \leqslant k < m_i\}$ of each H_i, then one can describe basic operators in $\mathcal{B}_{\mathrm{res}}(H)$ as follows: for each i and j, $i \in I$, $j \in I$, and all integers k and l, with $0 \leqslant k < m_i$ and $0 \leqslant l < m_j$, we define the operator $E_{(k,i)(l,j)}$ in $\mathcal{B}_{\mathrm{res}}(H)$ by

$$E_{(k,i)(l,j)}(e_t(s)) = \delta_{lt}\delta_{js}e_k(i).$$

The $\{E_{(k,i)(l,j)} \mid 0 \leqslant k < m_i \text{ and } 0 \leqslant l < m_j\}$ form a Hilbert basis of the space $\mathcal{HS}(H_j, H_i)$ of Hilbert–Schmidt operators from H_j to H_i.

Remark 2.8. The Lie algebra $\mathcal{B}_{\mathrm{res}}(H)$ varies with the decomposition (1) one considers. It is a significant enlargement of the Lie algebra $\mathcal{HS}(H, H)$ in order to have a wider range of generators for the flows that preserve the manifold. An example of this situation is the commuting flows for the KP-hierarchy from example (3.6).

As for graded properties of the Lie algebra $\mathcal{B}_{\mathrm{res}}(H)$, it possesses various Lie subalgebras such as loop algebras, e.g. that have a $\mathbb{Z}$-grading with finite dimensional homogeneous components, but none of them is dense in $\mathcal{B}_{\mathrm{res}}(H)$, so that they are all significantly smaller components of the algebra. This property can be seen from the fact that $\mathcal{B}_{\mathrm{res}}(H)$ contains, in general, subspaces that are the bounded linear operators on a Hilbert subspace of H. Decomposition (1) induces a $\mathbb{Z}$-grading only on a dense subalgebra of $\mathcal{B}_{\mathrm{res}}(H)$ and as such it does not fit into the category of Lie algebras as considered in [RS97] or [LS92]. Namely, if we introduce for $k \in I$, the subspace

$$\mathcal{B}_k := \{b = (b_{ij}) \in \mathcal{B}_{\mathrm{res}}(H), b_{ij} \text{ is non-zero only if } j = i + k\} \tag{3}$$

and, if we put $\mathcal{B}_l = 0$, for $l \in \mathbb{Z} - I$, then

$$\mathcal{B}_{fin} = \sum_{k \in \mathbb{Z}} \mathcal{B}_k \tag{4}$$

is a dense $\mathbb{Z}$-graded Lie subalgebra of $\mathcal{B}_{\mathrm{res}}(H)$.

The next step will be the introduction of the Lie group corresponding to $\mathcal{B}_{\mathrm{res}}(H)$. If $GL(H)$ denotes the group of invertible elements in $\mathcal{B}(H)$, then we have

Definition 2.9. *The restricted linear group, $GL_{\mathrm{res}}(H)$, consists of the group of invertible elements of $\mathcal{B}_{\mathrm{res}}(H)$.*

By definition $GL_{\mathrm{res}}(H)$ is a natural Banach Lie group with Lie algebra $\mathcal{B}_{\mathrm{res}}(H)$. From the two facts that the Hilbert–Schmidt operators form a two-sided ideal in $\mathcal{B}(H)$ and that a bounded operator is Hilbert–Schmidt as soon as the product of this operator with a Fredholm operator is Hilbert–Schmidt, one deduces that the group $GL_{\mathrm{res}}(H)$ is equal to $GL(H) \cap \mathcal{B}_{\mathrm{res}}(H)$.

Next we discuss a property of the group $GL_{\mathrm{res}}(H)$ if N, M or both are infinite. If $g \in GL_{\mathrm{res}}(H)$, then $g - \mathrm{diag}(g_{ii})$ is Hilbert–Schmidt. Hence it is compact and therefore $\mathrm{diag}(g_{ii})$ is a Fredholm operator of order zero. In particular, all the g_{ii} are Fredholm operators and if $M = \infty$, resp. $N = \infty$, then there is an i_0, resp. j_0, such that for all $i \geqslant i_0$, resp. all $j \leqslant j_0$, the operator g_{ii}, resp. g_{jj}, is invertible. For $M = \infty$ and i tending to infinity, the Hilbert–Schmidt norm of the off-diagonal part of $g_{\geqslant i}$ tends to zero and the same holds for the off-diagonal part of $g_{\leqslant j}$ for j tending to $-N - 1 = -\infty$. Since the operator norms of the $\{g_{ii} \mid \geqslant i_0\}$ and $\{g_{jj} \mid j \leqslant j_0\}$ are bounded away from zero, this leads to

Lemma 2.10. *Let $g \in GL_{\mathrm{res}}(H)$, where M, resp. N, is infinite. Then there exists an i_0, resp. j_0, such that for all $i \geqslant i_0$, resp. all $j \leqslant j_0$, the operator $g_{\geqslant i}$, resp. $g_{\leqslant j}$, is invertible.*

The analogue of the unitary group $U(H)$ in this context is:

Definition 2.11. *The restricted unitary group,* $U_{\mathrm{res}}(H) = GL_{\mathrm{res}}(H) \cap U(H)$.

Both $U_{\mathrm{res}}(H)$ and $GL_{\mathrm{res}}(H)$ are natural generalizations of the restricted unitary and general linear group, introduced in chapter 6 of [PS86]. If A^* denotes the adjoint of an operator $A \in \mathcal{B}(H)$, then the Lie algebra of $U_{\mathrm{res}}(H)$ consists of

$$\mathfrak{u}_{\mathrm{res}}(H) = \{X \mid X \in \mathcal{B}_{\mathrm{res}}(H),\ X^* = -X\}.$$

This is a real Lie subalgebra of $\mathcal{B}_{\mathrm{res}}(H)$ and the Lie algebra $\mathcal{B}_{\mathrm{res}}(H)$ can be written as

$$\mathcal{B}_{\mathrm{res}}(H) = \mathfrak{u}_{\mathrm{res}}(H) \oplus \mathrm{i}\mathfrak{u}_{\mathrm{res}}(H).$$

In other words $\mathcal{B}_{\mathrm{res}}(H)$ is the complexification of $\mathfrak{u}_{\mathrm{res}}(H)$. On the group level this corresponds to the fact that the group $GL_{\mathrm{res}}(H)$ possesses a 'polar decomposition' of which $U_{\mathrm{res}}(H)$ forms the unitary component. For, consider the sets

$$P(H) = \{A \mid A \in GL(H), A = A^* \quad \text{and} \quad A > 0\} \quad \text{and} \quad P_{\mathrm{res}}(H) = \mathcal{B}_{\mathrm{res}}(H) \cap P(H).$$

On $P_{\mathrm{res}}(H)$ we put the topology induced by $\mathcal{B}_{\mathrm{res}}(H)$. Since the map $A \mapsto \sqrt{A}$ from $P_{\mathrm{res}}(H)$ to $P(H)$ is locally given by a convergent power series in A, this map is in fact a continuous map from $P_{\mathrm{res}}(H)$ to itself. Thus we get

Proposition 2.12. *The map* $(u, p) \mapsto up$ *from* $U_{\mathrm{res}}(H) \times P_{\mathrm{res}}(H)$ *to* $GL_{\mathrm{res}}(H)$ *is a homeomorphism.*

Proof. The inverse of this map is

$$g \mapsto \left(g\sqrt{g^*g}^{-1}, \sqrt{g^*g}\right)$$

and we have just seen that it is continuous. □

With each $g \in GL_{\mathrm{res}}(H)$ we can associate the flag $gF^{(0)}$ given by

$$g(H(-N-1)) = 0 \subset \ldots \subset g\left(\bigoplus_{j \leqslant i} H_j\right) \subset \ldots \subset g(H(M)) = H.$$

From the way the group $GL_{\mathrm{res}}(H)$ is defined it is clear that this flag belongs to $\mathfrak{F}$.

The group $U_{\mathrm{res}}(H)$ acts already transitively on $\mathfrak{F}$. Let $F = \{F_i, i \in I\}$ belong to $\mathfrak{F}$. From the definition of $\mathfrak{F}$ we know that there is for each i, $i \in I$, an isometry u_i between H_i and F_i. If we put $u = \oplus_i u_i$, then the condition defining $\mathfrak{F}$ implies that u belongs to the group $U_{\mathrm{res}}(H)$ and that $F = u(F^0)$.

The stabilizer in $GL_{\mathrm{res}}(H)$ of the basic flag is the parabolic subgroup

$$P = \left\{ g = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & g_{mm} & \ldots & \ldots & g_{ml} & \ddots \\ \ddots & 0 & \ddots & & \vdots & \ddots \\ \ddots & \vdots & \ddots & \ddots & \vdots & \ddots \\ \ddots & 0 & \ldots & 0 & g_{ll} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \in GL_{\mathrm{res}}(H),\ \text{with}\, g_{ii} \in GL(H_i)\, \text{for all}\, i \right\}.$$

Since the group $U_{\mathrm{res}}(H)$ also acts transitively on $\mathfrak{F}$ and the stabilizer of the basic flag in this group is

$$P \cap U_{\mathrm{res}}(H) = \{u \mid u \in U_{\mathrm{res}}(H), u = \mathrm{diag}(u_{ii}),\ \text{with}\, u_{ii} \in U(H_i)\}$$

one can see $\mathfrak{F}$ both as $GL_{\mathrm{res}}(H)/P$ and as $U_{\mathrm{res}}(H)/P \cap U_{\mathrm{res}}(H)$.

Let $\tau : GL_{\mathrm{res}}(H) \to \mathfrak{F}$ be the projection $\tau(g) = gF^{(0)}$. On $\mathfrak{F}$ we will put a Hilbert manifold structure that makes τ into an open submersion. This will be discussed in the next section.

3. The manifold structure

In this section we discuss the Hilbert manifold structure on $\mathfrak{F}$ and some decompositions of open subsets in $GL_{\mathrm{res}}(H)$. From the definition of the parabolic group P one sees directly that the Lie algebra of P is given by

$$L(P) = \{g \mid g = (g_{ij}) \in B_{\mathrm{res}}(H),\ g_{ij} = 0 \text{ for all } i < j\}$$

and that a complement of $L(P)$ in $B_{\mathrm{res}}(H)$ is the Hilbert space $(E, \|\cdot\|_{\mathcal{HS}})$ with

$$E = \bigoplus_{\substack{i\in I\\ i<j}} \mathcal{HS}(H_j, H_i).$$

From section 6.1 in [Bou98], we know then that the homogeneous space $\mathfrak{F} = GL_{\mathrm{res}}(H)/P$ carries an analytic E-manifold structure for which τ is a submersion and for which the action L_g of $g \in GL_{\mathrm{res}}(H)$ on $\mathfrak{F}$ by left translation is analytic.

Next we give descriptions of some open subsets in $GL_{\mathrm{res}}(H)$ that will be needed later on and occur also at the construction of solutions of related integrable systems. Consider for each k, $k \in I$, the set $\Omega(\geqslant k)$ in $GL_{\mathrm{res}}(H)$ given by

$$\Omega(\geqslant k) = \left\{ g \in GL_{\mathrm{res}}(H) \mid g_{\geqslant i} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots \\ \ddots & g_{rr} & \cdots & g_{ri} \\ \ddots & \vdots & & \vdots \\ \ddots & g_{ir} & \cdots & g_{ii} \end{pmatrix} \in GL_{\mathrm{res}}(\oplus_{j\geqslant i} H_j) \quad \text{for all } i \geqslant k \right\}$$

and its counterpart $\Omega(\leqslant k)$ defined by

$$\Omega(\leqslant k) = \left\{ g \in GL_{\mathrm{res}}(H) \mid g_{\leqslant i} = \begin{pmatrix} g_{ii} & \cdots & g_{ir} & \ddots \\ \vdots & & \vdots & \ddots \\ g_{ri} & \cdots & g_{rr} & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \in GL_{\mathrm{res}}(\oplus_{j\leqslant i} H_j) \quad \text{for all } i \leqslant k \right\}.$$

Further we use the notation $\Omega_+(H)$, resp. $\Omega_-(H)$, for the intersection of all members of each collection, i.e.

$$\Omega_+(H) = \Omega_+ := \bigcap_{k\in I} \Omega(\geqslant k) \qquad \text{and} \qquad \Omega_-(H) = \Omega_- := \bigcap_{k\in I} \Omega(\leqslant k).$$

By using lemma 2.10 and the fact that the invertible operators are open in the space of Fredholm operators one sees that the sets $\Omega(\geqslant k)$, $\Omega(\leqslant k)$, Ω_+ and Ω_- are open and we discuss here their decomposition in terms of subgroups of $GL_{\mathrm{res}}(H)$. Inside this group we introduce the Lie subgroups $U_-(\geqslant k)$ and $P(\geqslant k)$ defined by, respectively,

$$U_-(\geqslant k) = \left\{ g = (g_{ij}) \in GL_{\text{res}}(H) \;\middle|\; \begin{array}{ll} g_{ii} = \text{Id}_{H_i} & \text{for all } i \\ g_{ij} = 0 & \text{for } i > j \\ g_{ij} = 0 & \text{for } i < j \text{ and } j < k \end{array} \right\}$$

$$P(\geqslant k) = \{g = (g_{ij}) \in GL_{\text{res}}(H) \mid g_{ij} = 0 \quad \text{if } i < j \text{ and } j \geqslant k\}.$$

If $u \in U_-(\geqslant k)$ and $p \in P(\geqslant k)$ and $g = up$, then it is a direct verification that for all $l \geqslant k$ there holds $g_{\geqslant l} = u_{\geqslant l}p_{\geqslant l}$, hence g belongs to $\Omega(\geqslant k)$. The union of all the groups $U_-(\geqslant k)$ is again a group and we denote it as follows:

$$U_- = U_-(H) = \bigcup_{k\in I} U_-(\geqslant k).$$

For the counterpart $\Omega(\leqslant k)$ we introduce similarly the groups $U_-(\leqslant k)$ and $P(\leqslant k)$ given by

$$U_-(\leqslant k) = \left\{ g = (g_{ij}) \in GL_{\text{res}}(H) \;\middle|\; \begin{array}{ll} g_{ii} = \text{Id}_{H_i} & \text{for all } i \\ g_{ij} = 0 & \text{for } j < i \\ g_{ij} = 0 & \text{for } i < j \text{ and } i > k \end{array} \right\}$$

and

$$P(\leqslant k) = \{g = (g_{ij}) \in GL_{\text{res}}(H) \mid g_{ij} = 0 \quad \text{if } i < j \text{ and } i \leqslant k\}.$$

If we take now a $u \in U_-(\leqslant k)$ and $p \in P(\leqslant k)$ and put $g = pu$, then there holds for all $l \leqslant k$ that $g_{\leqslant l} = p_{\leqslant l}u_{\leqslant l}$. Hence g belongs to $\Omega(\leqslant k)$. Clearly $P(\geqslant k) \cap U_-(\geqslant k) = \{\text{Id}_H\} = P(\leqslant k) \cap U_-(\leqslant k)$ and this gives you the injectivity of both the following maps.

Proposition 3.1. *The map $(u, p) \mapsto up$ from $U_-(\geqslant k) \times P(\geqslant k) \to GL_{\text{res}}(H)$ determines a homeomorphism between $U_-(\geqslant k) \times P(\geqslant k)$ and $\Omega(\geqslant k)$. Likewise, the map $(p, u) \mapsto pu$ from $P(\leqslant k) \times U_-(\leqslant k) \to GL_{\text{res}}(H)$ is a homeomorphism between $P(\leqslant k) \times U_-(\leqslant k)$ and $\Omega(\leqslant k)$.*

Proof. We merely have to reconstruct in an analytic way for an element $g \in \Omega(\geqslant k)$, resp. $\Omega(\leqslant k-1)$, the elements $u_1 \in U_-(\geqslant k)$ and $p_1 \in P(\geqslant k)$, resp. $u_2 \in U_-(\leqslant k-1)$ and $p_2 \in P(\leqslant k-1)$, such that $g = u_1 p_1$, resp. $g = p_2 u_2$. First we decompose these elements w.r.t. $H = (\oplus_{j\geqslant k} H_j) \oplus (\oplus_{j<k} H_j)$, respectively, as follows:

$$g = \begin{pmatrix} g_{\geqslant k} & g_>(k) \\ g_<(k) & g_{\leqslant k-1} \end{pmatrix} = \begin{pmatrix} Id & 0 \\ g_<(k)g_{\geqslant k}^{-1} & Id \end{pmatrix} \begin{pmatrix} g_{\geqslant k} & g_>(k) \\ 0 & g_{\leqslant k-1} - g_<(k)g_{\geqslant k}^{-1}g_>(k) \end{pmatrix}$$

$$g = \begin{pmatrix} g_{\geqslant k} & g_>(k) \\ g_<(k) & g_{\leqslant k-1} \end{pmatrix} = \begin{pmatrix} g_{\geqslant k} - g_>(k)g_{\leqslant k-1}^{-1}g_<(k) & g_>(k) \\ 0 & g_{\leqslant k-1} \end{pmatrix} \begin{pmatrix} Id & 0 \\ g_{\leqslant k-1}^{-1}g_<(k) & Id \end{pmatrix}.$$

This reduces the problem for $\Omega(\geqslant k)$ to decomposing an element $\tilde{g} := g_{\geqslant k} \in \Omega_+(\oplus_{j\geqslant k} H_j)$ as $\tilde{g} = \tilde{u}\tilde{p}$, where $\tilde{u} \in U_-(\oplus_{j\geqslant k} H_j)$ and $\tilde{p} \in P(\oplus_{j\geqslant k} H_j)$ and for $\Omega(\leqslant k-1)$ to the decomposition of an element $\hat{g} := g_{\leqslant k-1} \in \Omega_-(\oplus_{j<k} H_j)$ as $\hat{g} = \hat{p}\hat{u}$, where $\hat{u} \in U_-(\oplus_{j<k} H_j)$ and $\hat{p} \in P(\oplus_{j<k} H_j)$. Each of these decompositions is found by a step by step procedure, where we use the decomposition of $\tilde{g}$ w.r.t. $\oplus_{j\geqslant k} H_j = (\oplus_{j>k} H_j) \oplus H_k$ and that of $\hat{g}$ w.r.t. $\oplus_{j\leqslant k-1} H_j = H_{k-1} \oplus (\oplus_{j<k-1} H_j)$ given by

$$\tilde{g} = \begin{pmatrix} g_{\geqslant k+1} & g_+ \\ g_- & g_{kk} \end{pmatrix} = \begin{pmatrix} Id & 0 \\ g_- g_{\geqslant k+1}^{-1} & Id \end{pmatrix} \begin{pmatrix} g_{\geqslant k+1} & g_+ \\ 0 & g_{kk} - g_- g_{\geqslant k+1}^{-1} g_+ \end{pmatrix}$$

$$\hat{g} = \begin{pmatrix} g_{k-1k-1} & g_2 \\ g_1 & g_{\leqslant k-2} \end{pmatrix} = \begin{pmatrix} g_{kk} - g_2 g_{\leqslant k-2}^{-1} g_1 & g_2 \\ 0 & g_{\leqslant k-2} \end{pmatrix} \begin{pmatrix} Id & 0 \\ g_{\leqslant k-2}^{-1} g_1 & Id \end{pmatrix}.$$

Continuing in this fashion this procedure finishes for $\Omega(\geqslant k)$, resp. $\Omega(\leqslant k)$, in a finite number of steps if and only if $M < \infty$, resp. $N < \infty$, and yields then the required decomposition. In the infinite case the increasing product of elements in $U_-(\oplus_{j\geqslant k} H_j)$, resp. $U_-(\oplus_{j<k} H_j)$, converges to an element in these groups respectively, since the operator norms of the $\{g_{\geqslant l}, l \geqslant k\}$ and $\{g_{\leqslant l}, l < k\}$ are bounded away from zero and the sum over the Hilbert–Schmidt norms of the off-diagonal components is finite. This proves the lemma. □

Next we consider the open subsets Ω_+ and Ω_-. From the foregoing results it follows that $U_-P \subset \Omega_+$ and $PU_- \subset \Omega_-$. Note that we have already shown equality here in the cases that M or N is finite. So what remains is the case $M = N = \infty$. To show equality in this case, we take a $g \in \Omega_+$ and an $h \in \Omega_-$. Thanks to proposition 3.1 we may assume that $g \in P(\geqslant k)$ and $h \in P(\leqslant k-1)$. These operators are then reduced to the required form again by a step by step procedure. For, decompose both operators w.r.t. $H = (\oplus_{j\geqslant k} H_j) \oplus H_{k-1} \oplus (\oplus_{j<k-1} H_j)$, then we have respectively

$$g = \begin{pmatrix} g_{\geqslant k} & * & * \\ 0 & g_{k-1k-1} & * \\ 0 & p_4 & * \end{pmatrix} = \begin{pmatrix} Id & 0 & 0 \\ 0 & Id & 0 \\ 0 & p_4 g_{k-1k-1}^{-1} & Id \end{pmatrix} \begin{pmatrix} g_{\geqslant k} & * & * \\ 0 & g_{k-1k-1} & * \\ 0 & 0 & * \end{pmatrix}$$

$$h = \begin{pmatrix} * & * & * \\ q_2 & h_{k-1k-1} & * \\ 0 & 0 & h_{\leqslant k-2} \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & h_{k-1k-1} & * \\ 0 & 0 & h_{\leqslant k-2} \end{pmatrix} \begin{pmatrix} Id & 0 & 0 \\ h_{k-1k-1}^{-1} q_2 & Id & 0 \\ 0 & 0 & Id \end{pmatrix}.$$

Continuing in this fashion one finds all the components of the U_--component of g, resp. h, which are the limits of the increasing products of elements in U_- due to the step by step procedure. Their convergence is based on the same argument as above. We thus have found a Gauss-type decomposition for the sets Ω_+ and Ω_- and resum this result in a

Corollary 3.2. *Inside $GL_{\mathrm{res}}(H)$, we have $U_-P = \Omega_+$ and $PU_- = \Omega_-$. In particular we see that $\Omega_+ = \Omega_-^{-1}$.*

From this corollary it follows that the restriction of τ to U_- gives a diffeomorphism $u \mapsto uF^{(0)}$ between U_- and the open neighbourhood $\tau(\Omega_+)$ of $F^{(0)}$. Clearly the group U_- is diffeomorphic to the Hilbert space E. Note that from the definition of Ω_+ one can conclude directly that

$$\tau(\Omega_+) = \left\{ F = (F_i) \in \mathfrak{F} \;\middle|\; \bigoplus_{j\leqslant l} p_j : \bigoplus_{j\leqslant l} F_j \to \bigoplus_{j\leqslant l} H_j \quad \text{is a bijection for all } l \in I \right\}.$$

This characterization of $\tau(\Omega_+)$ tells you how to choose around a general point of $\mathfrak{F}$ a concrete neighbourhood diffeomorphic to E. This requires, however, the introduction of the following notation.

Notation 3.3. If W is a closed subspace of H, then we denote the orthogonal projection on W by p_W.

Consider a $F = (F_i)$ in $\mathfrak{F}$. Then the analogue of $\tau(\Omega)$ for F is

$$U_F = \left\{ V = (V_i) \text{ in } \mathfrak{F} \;\middle|\; \bigoplus_{i\leqslant l} p_{F_i} : \bigoplus_{i\leqslant l} V_i \to \bigoplus_{i\leqslant l} F_i \quad \text{is a bijection for all } l \in I \right\}.$$

Since $\mathfrak{F} = U_{\text{res}}(H)F^{(0)}$, we have for all F and G in $\mathfrak{F}$ that the map $\sum_j \sum_{i\neq j} p_{F_i} \mid G_j$ is a Hilbert–Schmidt operator. Hence, if V belongs to U_F, then there is a unique operator A in $\bigoplus_{\substack{i\in I\\ i<j}} \mathcal{HS}(F_j, F_i)$ such that for all i, $-N-1 \leqslant i \leqslant M+1$,

$$V(i) = \{x + A(x) \mid x \in F(i)\}.$$

This is why we call V also the graph of A and write $V = \text{graph}\,(A)$.

It is convenient to have a special name for the flags in U_F.

Definition 3.4. *A flag V in U_F is called transversal to F.*

Let g_F be an element of $U_{\text{res}}(H)$ such that $g_F F^{(0)} = F$. Instead of the big cell Ω_+ in $GL_{\text{res}}(H)$ with respect to the decomposition $H = \oplus_i H_i$, we could also have introduced a big cell with respect to $H = \oplus_i F_i$ and it will be clear that this set can be written as

$$g_F U_- P\big(g_F^{-1}\big).$$

Consequently, we get for U_F that

$$U_F = \big\{g_F u p (g_F)^{-1} F \mid \text{with } u \in U_- \text{ and } p \in P\big\} = \tau(g_F U_- P).$$

Then we can define for each F in $\mathfrak{F}$ a diffeomorphism $\varphi_F \colon U_F \to E$ by

$$\varphi_F\big(g_F u F^{(0)}\big) = u - Id.$$

Each (U_F, φ_F) is a concrete chart around F for the E-manifold structure on $\mathfrak{F}$.

We have now obtained a concrete description of the manifold structure on $\mathfrak{F}$:

Proposition 3.5. *The (U_F, φ_F) are the charts of the analytic E-manifold structure on $\mathfrak{F}$.*

Proof. It is sufficient to show for each $U_{F^{(1)}}$ and $U_{F^{(2)}}$ with $U_{F^{(1)}} \cap U_{F^{(2)}} \neq \emptyset$ that

$$\varphi_{F^{(2)}} \circ \varphi_{F^{(1)}}^{-1} : \varphi_{F^{(1)}}(U_{F^{(1)}} \cap U_{F^{(2)}}) \to \varphi_{F^{(2)}}(U_{F^{(1)}} \cap U_{F^{(2)}})$$

is an analytic map. From the step by step decomposition described in proposition 3.1 it follows that the U_--component of $(g_{F^{(2)}})^{-1} g_{F^{(1)}} u$ actually depends analytically on u. This proves the proposition. □

We conclude this subsection by describing the role the flag varieties corresponding to the example 2.2 play for the KP-hierarchy and its subsystems.

Example 3.6. Recall that the KP-hierarchy consists of a tower of nonlinear differential equations in infinitely many variables $\{t_n \mid n \geqslant 1\}$. It is named after the simplest nontrivial equation in this tower, the Kadomtsev–Petviashvili equation:

$$\frac{3}{4}\frac{\partial^2 u}{\partial t_2^2} = \frac{\partial}{\partial t_1}\left(\frac{\partial u}{\partial t_3} - 3u\frac{\partial u}{\partial t_1} - \frac{1}{4}\frac{\partial^3 u}{\partial t_1^3}\right) \tag{5}$$

which is a two-dimensional generalization of the KdV-equation. We consider solutions of these equations that belong to a commutative ring of functions R, which is stable under the operators $\partial_n = \frac{\partial}{\partial t_n}$. The compact form in which one usually presents the equations of the hierarchy, is the so-called Lax form. This is an equality between operators in the privileged derivation $\partial = \partial_1$ of a specific nature. This simple way to present the equations requires that one extends the ring $R[\partial] = \left\{\sum_{i=0}^n a_i \partial^i \,\middle|\, a_i \in R\right\}$ and adds suitable integral operators to the ring. Then it becomes possible to take the inverse and roots of certain differential operators. For example, the square root $\mathcal{L}^{\frac{1}{2}}$ of the Schrödinger operators $\mathcal{L} = \partial^2 + 2u$ is well defined in this extension. Thus one arrives at the ring $R[\partial, \partial^{-1})$ of pseudodifferential operators with coefficients in R. It consists of all expressions

$$\sum_{i=-\infty}^{N} a_i \partial^i \qquad a_i \in R \quad \text{for all } i$$

that are added in an obvious way and multiplied according to

$$\partial^j \circ a\partial^i = \sum_{k=0}^{\infty} \binom{j}{k} \partial^k(a)\partial^{i+j-k}.$$

Each operator $P = \sum p_j \partial^j$ decomposes as $P = P_+ + P_-$ with $P_+ = \sum_{j\geqslant 0} p_j \partial^j$ its differential operator part and $P_- = \sum_{j<0} p_j \partial^j$ its integral operator part. An operator $L \in R[\partial, \partial^{-1})$ of the form

$$L = \partial + \sum_{j<0} \ell_j \partial^j \qquad \ell_j \in R \quad \text{for all } j < 0 \tag{6}$$

carries the name *Lax operator*. We call a Lax operator a *solution of the KP hierarchy* if and only if it satisfies the system of equations

$$\partial_n(L) = \sum_{j<0} \partial_n(\ell_j)\partial^j = [(L^n)_+, L] \qquad n \geqslant 1. \tag{7}$$

They are called the *Lax equations* for L. As such they are a generalization of the so-called Lax equations of the mth Gelfand–Dickey hierarchy, which is the following system of equations for a differential operator $\mathcal{L} = \partial^m + \sum_{i\leqslant m-2} l_i \partial^i$ in $R[\partial]$,

$$\partial_n(\mathcal{L}) = \left[\left(\mathcal{L}^{\frac{n}{m}}\right)_+, \mathcal{L}\right] \qquad n \geqslant 1. \tag{8}$$

For example, for $m = 2$ this operator $\mathcal{L}$ will be the Schrödinger operator $\partial^2 + 2u$ and the case $n = 3$ of the Lax equations (8) is then equivalent to the property that u is a solution of the KdV-equation. A similar situation occurs in the KP-case. First one shows that the Lax equations of the KP-hierarchy are equivalent to the following infinite set of conditions for the Lax operator L:

$$\partial_n(L^m)_+ - \partial_m(L^n)_+ = [(L^n)_+, (L^m)_+] \qquad m, n \geqslant 1. \tag{9}$$

The case $n = 3$ and $m = 2$ of the system (9) implies then that the coefficient ℓ_{-1} of L is a solution of the KP-equation.

The equations of the KP-hierarchy possess a rich collection of solutions besides the trivial one $L = \partial$. In [SW85], they considered the Hilbert space from example 2.2 with the decomposition corresponding to $\underline{s} = (0)$. On the associated Grassmann manifold $Gr(H)$ acts the group of the commuting flows

$$\Gamma_+ = \left\{ \gamma(t) := \exp\left(\sum_{i\geqslant 1} t_i z^i\right) \middle| \, t_i \in \mathbb{C}, \sum_{i\geqslant 1} |t_i| N^i < \infty \text{ for some } N > 1 \right\}.$$

By transferring an arbitrary plane W in $Gr(H)$ with this group Γ_+ into the open cell Ω_+ and using the Gauss decomposition from corollary 3.2, they constructed for each W in $Gr(H)$ a solution L_W of the KP-hierarchy. We will refer to this set of solutions of the KP-hierarchy as the Segal–Wilson class. The coefficients of the Lax operators thus constructed turned out to be meromorphic functions on the group Γ_+. Besides the construction of this extensive class of solutions, Segal and Wilson also gave a geometric characterization of the solutions in $Gr(H)$ that are the mth root of a monic differential operator of order m, i.e. solutions of the

mth Gelfand–Dickey hierarchy. These are precisely the planes W satisfying $z^m W \subset W$. A geometric description of a generalization of this subsystem, the so-called vector constrained KP-hierarchy, was given in [HvdL98]. Similar constructions can be made for other groups of commuting flows (see, e.g., [HP93]).

Note that, if L is a Lax operator in $R[\partial, \partial^{-1})$, then for all monic P in $R[\partial]$ the operator PLP^{-1} is again a Lax operator. In view of the foregoing results a natural question is: given a solution L in $R[\partial, \partial^{-1})$ of the KP-hierarchy in the Segal–Wilson class, determine operators P in $R[\partial]$ such that $L_P = PLP^{-1}$ belong again to the Segal–Wilson class and describe these transformations geometrically in the context of the Grassmanian. This type of transformation for Schrödinger operators already occurred in the work of Darboux. Therefore these transformations and their inverses carry his name. The raised question was settled in [HvdL01a]. There it was shown that if V and W belong to $Gr(H)$ and V is a subspace of codimension n in W, then there is an explicit monic different operator P in $R[\partial]$ such that

$$L_V = PL_W P^{-1}.$$

In other words, the flag variety corresponding to $\underline{s} = (n, 0)$ describes the Darboux transformations of order n. Moreover, refinements of these flags, i.e. chains looking like the basic flag of type $\underline{s} = (n, \ldots s_i \ldots, 0)$, correspond to specific decompositions of the differential operator P. Similar questions can be raised for the subsystems mentioned above, we refer to [HvdL01a] for the answer.

4. The connected components of $GL_{\mathrm{res}}(H)$

Let $g = (g_{ij})$ be an element of $GL_{\mathrm{res}}(H)$. Recall that we have shown already that all diagonal components g_{ii} of g are Fredholm operators. The collection of Fredholm operators on a Hilbert space K is an open part of the space $\mathcal{B}(K)$. Its connected components are given by the index, which is defined as

$$\mathrm{ind}(B) = \dim(\ker(B)) - \dim(\mathrm{coker}(B))$$

where B is a Fredholm operator on K. Since the off-diagonal part of g is Hilbert–Schmidt and hence compact, the operator $\mathrm{diag}(g_{II})$ is a Fredholm operator of index zero. Moreover, we saw in lemma 2.10 that for sufficiently large $|i|$ the g_{ii} are invertible. Hence we have that the indices of the $\{g_{ii} \mid i \in I\}$ satisfy

$$\sum_{i\in I} \mathrm{ind}(g_{ii}) = 0 \qquad \text{and} \qquad \mathrm{ind}(g_{kk}) = 0 \qquad \text{if} \quad m_k < \infty \quad \text{or} \quad |k| \gg 0.$$

These relations lead to the introduction of the subgroup Z of $\mathbb{Z}^I$ defined by

$$Z = \left\{ z = (z_i) \in \mathbb{Z}^I \;\middle|\; \sum_{i\in I} z_i = 0, z_k = 0 \quad \text{if} \quad m_k < \infty \quad \text{or} \quad |k| \gg 0 \right\}.$$

On Z we take the discrete topology. The standard properties of the index imply that the map $i : GL_{\mathrm{res}}(H) \to Z$,

$$g \mapsto \{\mathrm{ind}(g_{ii})\}$$

is a continuous group homomorphism. Hence the sets

$$GL^{(z)}_{\mathrm{res}}(H) = \{g \mid g \in GL_{\mathrm{res}}(H), i(g) = z\} \qquad \text{with} \quad z \in Z$$

are open. In fact, they are exactly the connected components of $GL_{\mathrm{res}}(H)$, for

Proposition 4.1. *For each $z \in Z$, the set $GL^{(z)}_{\mathrm{res}}(H)$ is non-empty and connected.*

Proof. Let $z = (z_i)$ be in Z. To see that $GL_{\text{res}}^{(z)}(H)$ is non-empty, one has to consider only a finite number of the components $\{H_i\}$, since only a finite number of the $\{z_i\}$ are nonzero. That case has been shown in [HH94] and by extending the operator found there with the identity, one obtains an element in $GL_{\text{res}}^{(z)}(H)$. As for the connectedness it suffices to show that $GL_{\text{res}}^{(0)}(H)$ is connected. First one notes that, since P is homeomorphic to

$$\prod_{i\in I} GL(H_i) \times \prod_{j<i} \mathcal{HS}(H_j, H_i)$$

and all the $GL(H_i)$ are connected (see [Kui65]), the group P is connected. If we can show that each element of $GL_{\text{res}}^{(0)}(H)$ can be joined by a continuous path to an element of P, then this proves that $GL_{\text{res}}^{(0)}(H)$ is connected. Also this we reduce to the case of a finite number of components $\{H_i\}$, which has been treated in [HH94]. Assume $M = \infty$, if it were finite then one can proceed directly to the second step in the following reduction. For any element $g \in GL_{\text{res}}^{(0)}(H)$ we know from lemma 2.10 that there is a sufficiently large k_0 such that $g \in \Omega(\geqslant k_0)$. Hence $g = up \in U_-(\geqslant k_0)P(\geqslant k_0)$. Then the map $t \mapsto \{Id+(1-t)(u-Id)\}p$ joins g with p. If N is finite, then we have w.r.t. $H = (\oplus_{j\geqslant k_0} H_j) \oplus (\oplus_{j<k_0} H_j)$

$$p = \begin{pmatrix} p_{\geqslant k_0} & * \\ 0 & g_{\leqslant k_0-1} \end{pmatrix} \qquad \text{where} \quad g_{\leqslant k_0-1} \in GL_{\text{res}}^{(0)}(\oplus_{-N-1<j<k_0} H_j).$$

According to [HH94], the element $g_{\leqslant k_0-1}$ is linked by a continuous path with a $p(k_0) \in P(\oplus_{-N-1<j<k_0} H_j)$ and we are done. If $N = \infty$, then there is a sufficiently small l_0 such that $p \in \Omega(\leqslant l_0)$. Decomposing p w.r.t. $H = (\oplus_{j\geqslant k_0} H_j) \oplus (\oplus_{l_0<j<k_0} H_j) \oplus (\oplus_{j\leqslant l_0} H_j)$, results in

$$p = \begin{pmatrix} p_{\geqslant k_0} & * & * \\ 0 & g(0) & * \\ 0 & p(1) & p_{\leqslant l_0} \end{pmatrix} = \begin{pmatrix} p_{\geqslant k_0} & * & * \\ 0 & g(0) & * \\ 0 & 0 & p_{\leqslant l_0} \end{pmatrix} \begin{pmatrix} Id & 0 & 0 \\ 0 & Id & 0 \\ 0 & p_{\leqslant l_0}^{-1} p(1) & Id \end{pmatrix} =: p_1 u_1.$$

The map $t \mapsto p_1\{Id + (1-t)(u_1 - Id)\}$ links p with p_1. From proposition 3.1, we know that $p_{\leqslant l_0} = p(2)u(2)$, with $p(2) \in P(\oplus_{j\leqslant l_0} H_j)$ and $u(2) \in U_-(\oplus_{j\leqslant l_0} H_j)$. Hence we can decompose p_1 further as

$$p_1 = \begin{pmatrix} p_{\geqslant k_0} & * & * \\ 0 & g(0) & * \\ 0 & 0 & p_{\leqslant l_0} \end{pmatrix} = \begin{pmatrix} p_{\geqslant k_0} & * & * \\ 0 & g(0) & * \\ 0 & 0 & p(2) \end{pmatrix} \begin{pmatrix} Id & 0 & 0 \\ 0 & Id & 0 \\ 0 & 0 & u(2) \end{pmatrix} =: p_2 u_2.$$

By shrinking u_2 to the identity as we did before with similar operators, one links p_1 continuously with p_2. From [HH94] we know that $g(0)$ can be linked to a $p(0) \in P(\oplus_{l_0<j<k_0} H_j)$ and this concludes the proof of the proposition. □

This proposition is the extension to flag varieties of proposition 6.2.4 in [PS86].

Since the parabolic group P is connected, we see that

Corollary 4.2. *The connected components of* $\mathfrak{F}$ *are given by*

$$\mathfrak{F}^{(z)} = \left\{ gF^{(0)} \,\middle|\, g \in GL_{\text{res}}^{(z)}(H) \right\} \qquad z \in Z.$$

Remark 4.3. Let H be the Hilbert space from example 2.2. Then the connected components of $Gr(H)$ are labelled by the integers. It is clear that for each $m \in \mathbb{Z}$ and each $W \in Gr^{(n)}(H)$, the space $z^m W$ belongs to $Gr^{(n-m)}(H)$. The construction method in [SW85] of solutions of the KP-hierarchy is such that we have $L_W = L_{z^m W}$. Hence, for the set of solutions one could restrict to one component, but at Darboux considerations the other components are needed as well.

Example 4.4. Again we take the Hilbert space from example 2.2 and consider the Grassmannian $Gr(H)$. If we have for each $n \in \mathbb{Z}$ a plane $W_n \in Gr^{(n)}(H)$ and if these subspaces form an infinite flag

$$\cdots \subset W_{n-1} \subset W_n \subset W_{n+1} \subset W_{n-2} \subset \cdots \tag{10}$$

then we know from [HvdL01b] that this renders a solution of the 1-Toda lattice hierarchy. There it is also shown that an infinite chain

$$\cdots \subset W_{n-1} \subset W_n \subset W_{n+1} \subset \cdots W_0 = \left\{ \sum_{n \geqslant 0} a_n z^n \in H \right\}$$

yields systems of orthogonal polynomials occurring in matrix models. This illustrates that infinite flags are important as well.

5. The Fubini–Study metric

In this section we take the example of projective Hilbert space $\mathbb{P}^1(H)$, i.e. the manifold of all complex lines in H, and equip it with a Kähler structure that is the analogue of the finite dimensional case and for which its expression in local coordinates is not a notational disaster. Let H be

$$H = \left\{ \sum_{i \geqslant 0}^{\infty} \alpha_i e_i \;\middle|\; \sum_{i \geqslant 0}^{\infty} |\alpha_i|^2 < \infty \right\}. \tag{11}$$

As the orthogonal decomposition corresponding to $\mathbb{P}^1(H)$ we take

$$H = \{\alpha_0 e_0, \alpha_0 \in \mathbb{C}\} \oplus \left\{ \sum_{i > 0}^{\infty} \alpha_i e_i \in H \right\} := H_0 \oplus H_1. \tag{12}$$

Each $\sum_{i \geqslant 0}^{\infty} \alpha_i e_i \neq 0$ in H determines an element of $\mathbb{P}^1(H)$ that we denote by $[\alpha_i]$. The space $\mathbb{P}^1(H)$ is covered by the open subsets U_j, $j \geqslant 0$, given by

$$U_j = \{[\alpha_k], \alpha_j \neq 0\}.$$

The local complex coordinates on U_j are the $z_k(j) = x_k(i) + \mathrm{i} y_k(j) := \frac{\alpha_k}{\alpha_j}$, $k \neq j$ and instead of the real ones $x_k(j)$, $y_k(j)$, $k \neq j$, it is convenient to take the $z_k(j)$ and $\overline{z_k(j)}$, $k \neq j$. For $s \neq j$, we have on $U_j \cap U_s$ that

$$z_k(s) = \frac{z_k(j)}{z_s(j)} \qquad \text{for} \quad k \neq s \quad \text{and} \quad k \neq j \qquad z_j(s) = \frac{1}{z_s(j)}.$$

This defines the biholomorphic coordinate transformation $\varphi_{s,j}$ on $U_j \cap U_s$. On each U_j we have the vector fields

$$\frac{\partial}{\partial z_k(j)} := \frac{1}{2} \left(\frac{\partial}{\partial x_k(j)} - \mathrm{i} \frac{\partial}{\partial y_k(j)} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z_k(j)}} := \frac{1}{2} \left(\frac{\partial}{\partial x_k(j)} + \mathrm{i} \frac{\partial}{\partial y_k(j)} \right).$$

Hence any vector field X on U_j can be written as

$$X_u = \sum_{k \neq j} a_k(u) \frac{\partial}{\partial x_k(j)} + \mathrm{i} \sum_{k \neq j} b_k(u) \frac{\partial}{\partial y_k(j)} = \sum_{k \neq j} \alpha_k(u) \frac{\partial}{\partial z_k(j)} + \sum_{k \neq j} \overline{\alpha_k(u)} \frac{\partial}{\partial \overline{z_k(j)}} \tag{13}$$

where $\alpha_k(u) = a_k(u) + \mathrm{i} b_k(u)$. Since $\mathbb{P}^1(H)$ is a complex manifold, it has a canonical complex structure J that satisfies in each $u \in U_j$

$$J_u \left(\left. \frac{\partial}{\partial x_k(j)} \right|_u \right) = \left. \frac{\partial}{\partial y_k(j)} \right|_u \quad \text{and} \quad J_u \left(\left. \frac{\partial}{\partial y_k(j)} \right|_u \right) = - \left. \frac{\partial}{\partial x_k(j)} \right|_u. \tag{14}$$

We have seen that in each $u \in U_j$ the $\left\{\frac{\partial}{\partial z_k(j)}\Big|_u \text{ and } \frac{\partial}{\partial \overline{z_k(j)}}\Big|_u, k \neq j\right\}$ form a Hilbert basis of the tangent space at u considered as a real Hilbert space. Dual to these derivations are the elements $(\mathrm{d}z_k(j))_u$ and $(\mathrm{d}\overline{z_k(j)})_u$ in the cotangent space at u, giving rise to the sections $\mathrm{d}z_k(j)$, resp. $\mathrm{d}\overline{z_k(j)}$, of the complexified cotangent bundle $T^*\mathfrak{F} \otimes \mathbb{C}$ over U_j. The first span the subspace $T^{1,0}$ of complex-linear cotangent vectors and the second the space $T^{0,1}$ of complex-antilinear cotangent vectors. This splitting of the complex-valued 1-forms on U_j also extends to the complex-valued k-forms $\Omega^k(U_j, \mathbb{C})$ on U_j. They are a direct sum of the spaces $\Omega^{(l,m)}(U_j, \mathbb{C}), l + m = k$, of differential forms of type (l, m), which are by definition the sections of the $(\Lambda^l T^{1,0}) \wedge (\Lambda^m T^{0,1})$. By composing the exterior derivative d with the projections on the factors $(\Lambda^l T^{1,0}) \wedge (\Lambda^m T^{0,1})$, one gets the operators ∂ and $\bar{\partial}$

$$\partial : \Omega^{(l,m)}(U_j, \mathbb{C}) \to \Omega^{(l+1,m)}(U_j, \mathbb{C}) \quad \text{and} \quad \bar{\partial} : \Omega^{(l,m)}(U_j, \mathbb{C}) \to \Omega^{(l,m+1)}(U_j, \mathbb{C})$$

that are on the level of C^∞-functions given by

$$\partial(f) = \sum_{k \neq j} \frac{\partial}{\partial z_k(j)}(f)\, \mathrm{d}z_k(j) \qquad \text{and} \qquad \bar{\partial}(f) = \sum_{k \neq j} \frac{\partial}{\partial \overline{z_k(j)}}(f)\, \mathrm{d}\overline{z_k(j)}.$$

The sum of the operators ∂ and $\bar{\partial}$ is equal to the exterior derivative and they share with the exterior derivative the property $\partial^2 = 0 = \bar{\partial}^2$. Therefore they satisfy $\bar{\partial}\partial = -\partial\bar{\partial}$. On each U_j we have the function

$$K_j([\alpha_s]) := \ln\left(\sum_{k \neq j} |z_k(j)|^2 + 1\right) = \ln\left(\sum_{k \neq j} z_k(j)\overline{z_k(j)} + 1\right) =: \ln(V_j).$$

On the intersection $U_j \cap U_s$ these functions are linked by the relation

$$\varphi_{s,j}^* K_s = K_s \circ \varphi_{s,j} = K_j - \ln\left(|z_s(j)|^2\right) = K_j - \ln z_s(j) + \ln \overline{z_s(j)} \mod 2\pi \mathrm{i}\mathbb{Z}. \tag{15}$$

To K_j is associated the $(1, 1)$-form ω_j on U_j given by

$$\begin{aligned} \omega_j := \mathrm{i}\partial\bar{\partial}(K_j) &= \mathrm{i} \sum_{\substack{k \neq j \\ r \neq j}} \frac{\partial^2 K_j}{\partial z_k(j) \partial \overline{z_r(j)}} \mathrm{d}z_k(j) \wedge \mathrm{d}\overline{z_r(j)} \\ &= \mathrm{i} \sum_{\substack{k \neq j \\ r \neq j}} \frac{\delta_{kr} V_j - z_r(j)\overline{z_k(j)}}{V_j^2} \mathrm{d}z_k(j) \wedge \mathrm{d}\overline{z_r(j)}. \end{aligned} \tag{16}$$

Next we use the fact that the pullback by the holomorphic map $\varphi_{s,j}$ commutes with ∂ and $\bar{\partial}$ and we insert relation (15) to get

$$\varphi_{s,j}^* \omega_s = \mathrm{i}\partial\bar{\partial}\varphi_{s,j}^* K_s = \mathrm{i}\partial\bar{\partial} K_j = \omega_j. \tag{17}$$

Hence the $\{\omega_j\}$ compose to a global $(1, 1)$-form ω on $\mathbb{P}^1(H)$. The form ω is closed, for locally on U_j we have

$$\mathrm{d}\omega = (\partial + \bar{\partial})\omega = \mathrm{i}(\partial + \bar{\partial})(\omega_j) = \mathrm{i}(\partial + \bar{\partial})\partial\bar{\partial}K_j = 0 + \mathrm{i}\bar{\partial}\partial\bar{\partial}K_j = -\mathrm{i}\partial\bar{\partial}^2 K_j = 0.$$

From the local formula one deduces directly that for all vector fields X and Y, there holds $\omega(JX, JY) = \omega(X, Y)$, i.e. J is compatible with ω. With ω we also associate a metric g. For two vector fields X and Y it is given by $g(X, Y) = \omega(X, JY)$. To see that it is positive definite, we decompose a vector field X locally on U_j as in formula 13, we write $\alpha(u)$, resp. z, for the sequences $\{\alpha_k(u)\}$, resp. $\{z_k(j)\}$, and use the notation

$$\langle(\gamma_k), (\delta_k)\rangle_j := \sum_{k \neq j} \gamma_k \bar{\delta}_k$$

for square summable sequences. Then we get that for a nonzero vector field X

$$\omega(X, JX) = \sum_{\substack{k\neq j\\ r\neq j}} \frac{\delta_{kr} V_j - z_r(j)\overline{z_k(j)}}{V_j^2} \alpha_k(u)\overline{\alpha_r(u)}\, \mathrm{d}z_k(j) \wedge \mathrm{d}\overline{z_r(j)}$$
$$= \langle \alpha(u), \alpha(u)\rangle_j (1 + \langle z, z\rangle_j) - \langle \alpha(u), z\rangle_j \langle z, \alpha(u)\rangle_j > 0 \tag{18}$$

because of the Cauchy–Schwarz inequality. This metric we call, analogously to the finite dimensional case, the *Fubini–Study metric* on projective Hilbert space, Thus we have put a Kähler structure on $\mathbb{P}^1(H)$ and one way to proceed in the general case, would be to make a complex embedding of $\mathfrak{F}$ into some $\mathbb{P}^1(\mathcal{H})$. This would require the construction of holomorphic line bundles over $\mathfrak{F}$ with sufficiently many global sections. This approach is too cumbersome and we will present a more direct one in the next section. We conclude with an example from the KP-theory, where these homogeneous coordinates in a Hilbert space play a role.

Example 5.1. For the Hilbert space from example 2.2 we consider the Grassmannian $Gr(H)$. For $n \in \mathbb{Z}$, let $\Sigma(n) = \{m \in \mathbb{Z}, m \geqslant -n\}$. If $H_{\Sigma(n)}$ is the Hilbert span of the $\{z^s \mid s \in \Sigma(n)\}$, then it belongs to the component $Gr^{(n)}(H)$. Other basic examples in this component of $Gr(H)$ are the subspaces corresponding to subsets of $\mathbb{Z}$ comparable to $\Sigma(n)$. More precisely, consider

$$\mathcal{S}(n) = \{\Sigma = (s_i) \mid s_i \in \mathbb{Z}, i \geqslant -n, s_{i+1} > s_i, s_i = i \text{ for } i \gg 0\} \tag{19}$$

and let H_Σ be the Hilbert span of all the z^s, $s \in \Sigma$. All these spaces belong to $Gr^{(n)}(H)$. For each $\Sigma = (s_i)$ from $\mathcal{S}(n)$ we denote the orthogonal projection of H onto H_Σ by p_Σ and write σ_Σ for the isomorphism between $H_{\Sigma(n)}$ and H_Σ determined by $\sigma_\Sigma(z^i) = z^{s_i}, i \geqslant -n$. From the way $Gr(H)$ is defined, one deduces that any plane W in $Gr^{(n)}(H)$ is the image of an embedding $w : H_{\Sigma(n)} \mapsto H$, satisfying two properties. The first is that the component $w_+ := p_{\Sigma(n)} \circ w$ is of the form 'identity + trace class' so that $\det(w_+)$ is well defined. The second property is that the other component $w_- := \left(Id - p_{\Sigma(n)}\right) \circ w$ is a Hilbert–Schmidt operator. We denote the space of this type of embeddings by $\mathcal{P}_n$. Two embeddings w_1 and w_2 in $\mathcal{P}_n$ have the same image if and only if $w_1 = w_2 \circ t$ with t in the group

$$\mathcal{T}_n = \{t \in Aut(H_{\Sigma(n)}) \mid t - Id \text{ is trace class}\}. \tag{20}$$

Consider the product space $\mathcal{P}_n \times \mathbb{C}$ and the equivalence relation on it given by

$$(w_1, \lambda_1) \sim (w_2, \lambda_2) \Leftrightarrow w_1 = w_2 \circ t \qquad \text{with} \quad t \in \mathcal{T}_n \quad \text{and} \quad \lambda_1 = \lambda_2 \det(t).$$

We denote the equivalence class of (w, λ) by $[w, \lambda]$. The quotient under this equivalence relation defines the line bundle Det^* over $Gr^{(n)}(H)$. Note that for each $\Sigma = (s_i) \in \mathcal{S}(n)$ we have the holomorphic section

$$\text{Image}(w) \mapsto \left[w, \det\left(\sigma_\Sigma^{-1} \circ p_\Sigma \circ w\right)\right]$$

of this bundle. Now for each v and $w \in \mathcal{P}_n$ the operator $v^* \circ w - Id$ is equal to $(v_+)^* w_+ + (v_-)^* w_- - Id$ and thus is a trace-class operator. For the determinant of this product of a $\mathbb{Z} \times \mathcal{S}(n)$-matrix and a $\mathcal{S}(n) \times \mathbb{Z}$-matrix we have a formula similar to that for the product of an $k \times m$-matrix and a $m \times k$-matrix. There holds namely

$$\det(v^* \circ w) = \sum_{\Sigma \in \mathcal{S}(n)} \det(v^*|_{H_\Sigma} \circ \sigma_\Sigma) \det\left(\sigma_\Sigma^{-1} \circ p_\Sigma \circ w\right).$$

It can be derived by approximating $w_+ - Id$ and w_- with operators with a finite dimensional range. In particular, one sees that the sequence $\left(\det\left(\sigma_\Sigma^{-1} \circ p_\Sigma \circ w\right)\right)_{\Sigma \in \mathcal{S}(n)}$ belongs to $\ell^2(\mathcal{S}(n))$ and then one verifies that the map

$$W = \text{Image of } w \mapsto \left[\left(\det\left(\sigma_\Sigma^{-1} \circ p_\Sigma \circ w\right)\right)\right]$$

defines an embedding of $Gr^{(n)}(H)$ into the projective space of the Hilbert space $\ell^2(\mathcal{S}(n))$. The homogeneous coordinates $\big(\det\big(\sigma_\Sigma^{-1}\circ p_\Sigma\circ w\big)\big)_{\Sigma\in\mathcal{S}(n)}$ are called the *Plücker coordinates* of the plane W, as they are the analogue in the present setting of the finite dimensional notion.

The Plücker coordinates also turn up for the so-called τ-functions for the KP-hierarchy. Solutions L of the KP-hierarchy are usually meromorphic expressions in a single τ-function and its derivatives w.r.t. the privileged variable t_1 and these functions are determined up to a constant by L. For the solutions L_W, $W\in Gr(H)$, we can describe them concretely. The group Γ_+ from example 3.6 acts on each component $Gr^{(n)}(H)$. If we decompose the action of $\gamma(t)^{-1}$ w.r.t. $H=H_{\Sigma(n)}\oplus H_{\Sigma(n)}^{\perp}$ as $\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right)$, then the τ-function associated with $w\in\mathcal{P}_n$ is given by $\tau_w(\gamma(t)) := \det(w_+ + a^{-1}bw_-)$. The same formula for the determinant as above gives the decomposition of τ_w

$$\tau_w(\gamma(t)) := \det(w_+ + a^{-1}bw_-) = \sum_{\Sigma\in\mathcal{S}(n)} \det\big(\sigma_\Sigma^{-1}\circ p_\Sigma\circ w\big)\,\tau_{\sigma_\Sigma}(\gamma(t)) \qquad (21)$$

in the τ-functions of the planes H_Σ, $\Sigma\in\mathcal{S}(n)$. As shown in [SW85], these are the Schur functions corresponding to the partition $(i-s_i)_{i\geqslant -n}$.

6. The Kähler structure on $\mathfrak{F}$

Now that we know the manifold structure on $\mathfrak{F}$ we can discuss its tangent bundle and put a global Hermitian structure on it. Its imaginary part renders a global 2-form on $\mathfrak{F}$ that can be shown to be closed. Thus we will obtain the Kähler structure on $\mathfrak{F}$. Hence, here the positive definiteness is automatic and the closedness requires a proof, while it was the other way around in the foregoing section.

Since the exponential map is an analytic isomorphism from E to the group $U_-(H)$, we also have the isomorphism

$$Y \mapsto \exp(Y)F^{(0)}$$

between E and the neighbourhood $\tau(\Omega_+)$ of $F^{(0)}$. In particular, we see that the small perturbation of this map

$$Y \mapsto \exp(Y-Y^*)F^{(0)} = \{\exp(Y-Y^*)\exp(Y^*)\exp(-Y)\}\exp(Y)F^{(0)}$$

is an isomorphism from a neighbourhood of 0 in E to a neighbourhood $V^{(0)}$ of $F^{(0)}$ in $\mathfrak{F}$. Moreover, the tangent space in $F^{(0)}$ can be identified with E by associating with each $X\in E$ the derivation

$$f \mapsto X_{F^{(0)}}(f) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\big(\exp(tX)F^{(0)}\big)$$

where f stands for an arbitrary C^∞-function around $F^{(0)}$. For each $F=g_F F^{(0)}$ in $\mathfrak{F}$, with $g_F\in U_{\mathrm{res}}(H)$, a trivialization of the tangent bundle $T\mathfrak{F}$ on the open set $g_F V^{(0)}$ is given by

$$(g_F\exp(Y-Y^*), X) \mapsto d\big(L_{g_F\exp(Y-Y^*)}\big)\big(F^{(0)}\big)X_{F^{(0)}}. \qquad (22)$$

Since each $\exp(Y-Y^*)$, $Y\in E$, belongs to $U_{\mathrm{res}}(H)$, we note that if we choose different trivializations and have $g_{F_1}\exp(Y_1-Y_1^*)F^{(0)} = g_{F_2}\exp(Y_2-Y_2^*)F^{(0)}$, then there holds

$$g_{F_1}\exp(Y_1-Y_1^*) = g_{F_2}\exp(Y_2-Y_2^*)u \qquad \text{with} \quad u=\mathrm{diag}(u_{ii})$$

with each $u_{ii}\in U(H_i)$. Hence another choice of trivialization of the form 22 induces merely the following transformation on E:

$$X \mapsto uXu^{-1} \qquad \text{with} \quad u\in U_{\mathrm{res}}(H)\cap P.$$

Consider now any point $F = g_F F^{(0)}$ in $\mathfrak{F}$ and two arbitrary elements X_F and Y_F in the tangent space $T\mathfrak{F}_F$. They can, once that g_F has been chosen, uniquely be written as $X_F = d(L_{g_F})\big(F^{(0)}\big)X$ and $Y_F = d(L_{g_F})\big(F^{(0)}\big)Y$ with X and $Y \in E$. However, there is on the tangent space $T\mathfrak{F}_F$ a Hermitian form in which this dependence of g_F does not play a role anymore. Recall that the product of two Hilbert–Schmidt operators is an operator of trace class. Then we can define on $T\mathfrak{F}_F$ the form

$$B_F(X_F, Y_F) := \mathrm{trace}(Y^*X) = \sum_{i\in I} \mathrm{trace}\left(\sum_{k\in I}(Y^*)_{ik}X_{ki}\right) \tag{23}$$

$$= \sum_{i\in I}\sum_{k<i} \mathrm{trace}((Y^*)_{ik}X_{ki}) \tag{24}$$

since $Y^*_{ik} = 0$, unless $k < i$. The form B_F is independent of the choice of g_F, since we have for each $u \in U_{\mathrm{res}}(H) \cap P$ the relation

$$\mathrm{trace}((uYu^{-1})^*(uXu^{-1})) = \mathrm{trace}(uY^*Xu^{-1}) = \mathrm{trace}(Y^*X)$$

and it is clearly Hermitian. Note that this form is constant on $\mathfrak{F}$. Thus we have put a Hermitian structure on $T\mathfrak{F}_F$. The associated strong Riemannian metric is obtained by taking the real part of B_F

$$g(X_F, Y_F) = \tfrac{1}{2}(B_F(X_F, Y_F) + B_F(Y_F, X_F)) \tag{25}$$

$$= \sum_{i\in I}\sum_{k<i} \tfrac{1}{2}(\mathrm{trace}((Y^*)_{ik}X_{ki}) + \mathrm{trace}((X^*)_{ik}Y_{ki})). \tag{26}$$

Since B_F is Hermitian, its imaginary part defines an antisymmetric form Φ_F on $T\mathfrak{F}_F$ that is equal to

$$\Phi(X_F, Y_F) := g(X_F, \mathrm{i}Y_F) \tag{27}$$

$$= \sum_{i\in I}\sum_{k<i} \frac{1}{2\mathrm{i}}(\mathrm{trace}((Y^*)_{ik}X_{ki}) - \mathrm{trace}((X^*)_{ik}Y_{ki})). \tag{28}$$

The forms Φ_F determine the so-called *fundamental 2-form* Φ on $\mathfrak{F}$ by

$$\Phi(X_1, X_2)(F) := \Phi_F((X_1)_F, (X_2)_F)$$

for all vector fields X_1 and X_2 on $\mathfrak{F}$. It is well known, see e.g. [AMR88], that on a Hilbert manifold as $\mathfrak{F}$ the exterior derivative of an k-form ω is well defined and is given by

$$\mathrm{d}\omega(X_0, \ldots, X_k) = \sum_{l=O}^{k}(-1)^l \mathfrak{L}_{X_l}(\omega(X_0, \ldots, \hat{X}_l, \ldots, X_k)) \tag{29}$$

$$+ \sum_{o\leqslant i<j\leqslant k} (-1)^{i+j}\omega(\mathfrak{L}_{X_i}(X_j), X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)). \tag{30}$$

Here $\hat{X}_i$ denotes that X_i is deleted. Further, the $X_0, \ldots, X_k$ are vector fields on $\mathfrak{F}$ and $\mathfrak{L}_{X_i}$ denotes the Lie derivative w.r.t. X_i.

The manifold $\mathfrak{F}$ is called a *Kähler manifold* if the 2-form Φ is closed, i.e. $\mathrm{d}\Phi = 0$. The form Φ is also called a *Kähler form* then. Since the 2-form Φ is constant on $\mathfrak{F}$, all the expressions $\mathfrak{L}_{X_l}(\Phi(X_i, X_j))$ in the exterior derivative of Φ are zero. For the remaining expressions in $\mathrm{d}\Phi$, we use again that Φ is completely determined by $\Phi_{F^{(0)}}$ and obtain then from (29) the following condition for the closedness of Φ.

Proposition 6.1. *The 2-form Φ is closed if and only if $\Phi_{F^{(0)}}$ is a Lie algebra 2-cocycle, that is to say it satisfies for all X, Y and $Z \in E$ the relation*

$$\Phi_{F^{(0)}}(X, [Y, Z]) + \Phi_{F^{(0)}}(Y, [Z, X]) + \Phi_{F^{(0)}}(Z, [X, Y]) = 0. \tag{31}$$

Note that for $M + N = 1$, i.e. in the case of the Grassmannian, the commutators in the cocycle condition are all zero, so that the statement is trivially true. To show that $\Phi_{F^{(0)}}$ possesses the cocycle property, we prove the relation for X, Y and $Z \in E$ equal to any of the basic operators $E_{(k,i)(l,j)}$ in E and , since they form a Hilbert basis of E, the fact that $\Phi_{F^{(0)}}$ is continuous gives the general result. Let $X = E_{(k_1,i_1)(l_1,j_1)}$, $Y = E_{(k_2,i_2)(l_2,j_2)}$ and $Z = E_{(k_3,i_3)(l_3,j_3)}$, then there holds for $l = 1, 2$ and 3 that $i_l < j_l$. This restricts the possible outcomes of the commutator of Y and Z to

$$[Y, Z] = 0 \quad \text{if} \quad (l_2, j_2) \neq (k_3, i_3) \quad \text{and} \quad [Y, Z] = E_{(k_2,i_2)(l_3,j_3)} \quad \text{if} \quad (l_2, j_2) = (k_3, i_3).$$

If $(l_2, j_2) = (k_3, i_3)$, then we get

$$\Phi_{F^{(0)}}(X, [Y, Z]) = \frac{1}{2\mathrm{i}} \operatorname{trace}\bigl(E_{(l_3,j_3)(k_2,i_2)} E_{(k_1,i_1)(l_1,j_1)} - E_{(l_1,j_1)(k_1,i_1)} E_{(k_2,i_2)(l_3,j_3)}\bigr).$$

For $(k_2, i_2) = (k_1, i_1)$, there holds either $\operatorname{trace}\bigl(E_{(l_3,j_3)(l_1,j_1)}\bigr) = \operatorname{trace}\bigl(E_{(l_1,j_1)(l_3,j_3)}\bigr) = 0$ or $E_{(l_3,j_3)(l_1,j_1)} = E_{(l_1,j_1)(l_3,j_3)}$, so that $\Phi_{F^{(0)}}(X, [Y, Z]) = 0$ for these X, Y and $Z \in E$. The same reasoning holds for the other terms and thus we have obtained the final result.

Theorem 6.2. *The 2-form Φ is closed and $\mathfrak{F}$ is a Kähler manifold.*

Since $\mathfrak{F}$ is modelled on a Hilbert space the 2-form Φ is a symplectic form. As the Hermitian structure on $T\mathfrak{F}$ is invariant under left translations from $U_{\mathrm{res}}(H)$, we see that

Corollary 6.3. *The group $U_{\mathrm{res}}(H)$ acts by symplectomorphisms on the symplectic variety $\mathfrak{F}$.*

This result can be applied at Hamiltonian aspects of KP- and Toda-type integrable systems.

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